Analysis of Shells of Revolution Having Arbitrary Stiffness Distributions

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A general procedure is presented for the analysis of nonsymmetric shells of revolution subjected to arbitrary loads and temperatures. The analytical formulation is based on a modified form of Sanders' first-order linear shell theory, with the effects of shear distortion and orthotropic material behavior included. A Fourier approach in the circumferential coordinate is used to separate variables of the modified form of governing equations. The resulting coupled ordinary differential equations relating the unknown Fourier coefficients in the displacement and rotation variables are determined by the matrix solution of the finite-difference form. The numerical analysis was programed for solution on a digital computer, and sample numerical results are presented.

Nomenclature

s,θ,ζ = shell coordinates
r = normal distance from the shell axis
ξ = nondimensional meriodional shell coordinate
a,h_0,σ_0,E_0 = reference length, thickness, stress, and Young's modulus
R_ξ,R_θ = principal radii of curvature
$\omega_\theta,\omega_\xi,\rho,\gamma$ = nondimensional curvatures, Eq. (1)
$N_\xi,N_\theta,N_{\xi\theta}$ = membrane forces per unit length
$M_\xi,M_\theta,M_{\xi\theta}$ = bending moments per unit length
$\bar{N}_{\xi\theta}\bar{M}_{\xi\theta}$ = modified membrane shear and twisting moments
Q_ξ,Q_θ = transverse shear forces per unit length
q_ξ,q_θ,q_ζ = shell loads per unit area
σ_ξ,σ_θ = meridional and circumferential stresses
$\tau_{\xi\theta},\tau_{\xi\zeta},\tau_{\theta\zeta}$ = in-plane and transverse shear stresses
$\epsilon_\xi,\epsilon_\theta,\epsilon_{\xi\theta}$ = membrane strains
$\gamma_{\xi\zeta},\gamma_{\theta\zeta}$ = transverse shear strains
$\kappa_\xi,\kappa_\theta,\kappa_{\xi\theta}$ = bending distortion terms
E_ξ,E_θ = Young's moduli
$G_{\xi\theta},G_{\xi\zeta},G_{\theta\zeta}$ = moduli of shear
ν_ξ,ν_θ = Poisson's ratios
α_ξ,α_θ = coefficients of thermal expansion
T = temperature differential
B_{mr} = form of the stiffness parameters, Eq. (6)
Θ_{mr} = form of the thermal loads and moments, Eq. (7)
U_ζ,U_θ,W = meridional, circumferential, and normal displacements
Φ_ξ,Φ_θ = rotations
$b_j^{(mr)}$ = Fourier coefficients for stiffness parameters
$u,v,w,\phi,\bar{\phi}$ = Fourier coefficients for displacements and rotations
p_ξ,p_θ,p_ζ = Fourier coefficients for loads
t_{mn}^T, $\bar{t}_{mn}^T$ = Fourier cofficients for thermal loads and moments
A_{mr} = stiffness recursion relationship
K = number of terms retained in Fourier series expansions
Δ = finite-difference increments (units of ξ)

Matrices

F,G,H,R,S,A,B,C,P = $5K \times 5K$ matrices
p,l,t^*,g,x,z = $1 \times 5K$ column matrices
Ω,Λ = $5K \times 5K$ diagonal matrices

Indices

m,r = subscript and superscript indices
j = Fourier component stiffness expansion
n = Fourier component for load and solution
k = index $0,1,2,\ldots, K-1$
1 = dummy index set from 1 to 5
i = meridional station index

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Introduction

IN several recent investigations,[1–3] analyses have been presented for treating the problem of thin shells of revolution subjected to arbitrary loads. The basic analytical technique used involved uncoupling of the equations in the circumferential coordinate by the use of Fourier series. This technique is no longer applicable to shell-of-revolution configurations with varying material properties and wall thicknesses in the meridional and circumferential directions.

This paper presents a general procedure for analysis of an extended class of shells of revolution which have arbitrary stiffness distribution and are subjected to arbitrary loads and temperatures. The analytical formulation is based on a modified form of Sanders' linear first-order shell theory.[4] The applicability of this analysis is limited to shells that have a surface-of-revolution reference surface selected within or in proximity to the shell surfaces. Thus, the analysis is applicable as long as the variations of the physical shell in the circumferential direction do not force the selection of a reference surface that would cause significant changes in key geometric parameters.

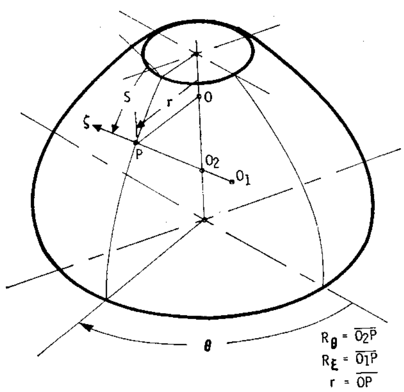

Fig. 1 Geometry and coordinates.

In the formulation, the effects of transverse shear distortion and orthotropic material behavior are included. The material properties and temperature distributions are permitted to vary across the shell thickness as well as in the surface coordinates. Thermal strain effects on material properties can be considered.

A Fourier analysis technique is used to separate variables. Substitution of the Fourier expansions into the constitutive relations and the use of appropriate trigonometric identities yield series expressions that relate forces and moments in terms of the Fourier coefficients of the displacement (rotation) variables and stiffnesses. Insertion of these expressions into Sanders' equilibrium equations and use of the appropriate orthogonality conditions yield a set of $5K$ unknown Fourier coefficients. (K is the finite number of Fourier terms retained in the assumed series solution.)

The Fourier components of the pertinent variables can then be found by the matrix solution of the finite-difference forms of the appropriate differential equations in the meridional coordinate. A direct matrix elimination technique (generalized Potters' method[5]) used successfully in Ref. 1 is used to solve the resulting sets of algebraic equations. The stresses, forces, etc. at any point in the shell can be determined from the knowledge of these displacements and rotations.

The numerical procedure was programed for solution on a digital computer. Representative numerical results are presented.

Analytical Formulation

Shell Geometry

The geometry of a shell is defined entirely by specifying the form of the reference surface and the thickness of the shell at each point. For convenience, the shell coordinate system and geometrical relations used in Ref. 1 will be adopted here. The reference surface is assumed to be a surface of revolution and is selected at a convenient location, within or in proximity to the shell walls. The first fundamental form of the reference surface is defined by $d\bar{s}^2 = ds^2 + r^2 d\theta^2$ where $d\bar{s}$ is a line element on the surface and s and θ denote orthogonal coordinates selected along lines of principal curvature. The generator of the reference surface is defined by $r(s)$ where r is the distance from the axis of revolution. The coordinate s is a measure of the meridional distance along an axisymmetric reference surface, and θ is a circumferential angle, as shown in Fig. 1. The coordinate ζ is selected as a measure of the normal outward distance from the reference surface ($\zeta = 0$). The principal radii of curvature are denoted by R_ξ and R_θ. From Ref. 1, the following nondimensional parameters are introduced:

$$\xi = s/a,\ \omega_\xi = a/R_\xi,\ \omega_\theta = a/R_\theta \qquad \rho = r/a,\ \gamma = (1/\rho)(\partial\rho/\partial\xi) \tag{1}$$

where a represents a reference length.

Equilibrium Equations

The general equilibrium equations for an arbitrary shell based upon the first-order linear theory of Sanders are given in Ref. 4. These equations, specialized for a shell which has as a reference surface a surface of revolution, are given by

$$a\left[\frac{\partial\rho}{\partial\xi}N_\xi + \rho\frac{\partial N_\xi}{\partial\xi} + \frac{\partial\bar{N}_{\xi\theta}}{\partial\theta} - \frac{\partial\rho}{\partial\xi}N_\theta\right] + a\rho\omega_\xi Q_\xi + \frac{1}{2}(\omega_\xi - \omega_\theta)\frac{\partial\bar{M}_{\xi\theta}}{\partial\theta} + a^2\rho q_\xi = 0 \tag{2a}$$

$$a\left[\frac{\partial N_\theta}{\partial\theta} + 2\frac{\partial\rho}{\partial\xi}\bar{N}_{\xi\theta} + \rho\frac{\partial\bar{N}_{\xi\theta}}{\partial\xi}\right] + a\rho\omega_\theta Q_\theta + \frac{\rho}{2}\frac{\partial}{\partial\xi}[(\omega_\theta - \omega_\xi)\bar{M}_{\xi\theta}] + a^2\rho q_\theta = 0 \tag{2b}$$

$$a\left[\frac{\partial\rho}{\partial\xi}Q_\xi + \rho\frac{\partial Q_\xi}{\partial\xi} + \frac{\partial Q_\theta}{\partial\theta} - \rho(\omega_\xi N_\xi + \omega_\theta N_\theta)\right] + a^2\rho q_\zeta = 0 \tag{2c}$$

$$\frac{\partial\rho}{\partial\xi}M_\xi + \rho\frac{\partial M_\xi}{\partial\xi} + \frac{\partial\bar{M}_{\xi\theta}}{\partial\theta} - \frac{\partial\rho}{\partial\xi}M_\theta - a\rho Q_\xi = 0 \tag{2d}$$

$$\frac{\partial M_\theta}{\partial\theta} + 2\frac{\partial\rho}{\partial\xi}\bar{M}_{\xi\theta} + \rho\frac{\partial\bar{M}_{\xi\theta}}{\partial\xi} - a\rho Q_\theta = 0 \tag{2e}$$

The ten stress resultants and couples acting on sections of the shell parallel to coordinate lines are shown in Fig. 2. Recall from Ref. 4 that $\bar{N}_{\xi\theta}$, $\bar{M}_{\xi\theta}$ represent modified variables formed by the appropriate combination of in-plane shear forces and twisting moments. It is noteworthy that, in the Sanders' formulation, the transverse shear strains were assumed to vanish in order to obtain these modified variables. Since transverse shear strain will be included in this study, it will be necessary to relax this restriction. As a result, the potential energy expression [Eq. (26), Ref. 4] takes a more complicated form due to additional terms resulting from the shear strains. If terms of the form (M_{12}–M_{21}) are neglected in the potential energy expression, however, it can be shown that the equilibrium equations, including the transverse shear distortion effects, reduce to the form given in Eqs. (2a–2e). A similar simplification was suggested by Sanders.

Constitutive Relationships

The forces and moments in equilibrium equations, Eqs. (2), are determined approximately by evaluating the following integrals through the thickness of the shell:

$$\begin{aligned}\{N_\xi,N_\theta,N_{\xi\theta}\} &= \int\{\sigma_\xi,\sigma_\theta,\tau_{\xi\theta}\}d\zeta\\ \{Q_\xi,Q_\theta\} &= \int\{\tau_{\xi\zeta},\tau_{\theta\zeta}\}d\zeta\\ \{M_\xi,M_\theta,\bar{M}_{\xi\theta}\} &= \int\{\sigma_\xi,\sigma_\theta,\tau_{\xi\theta}\}\zeta d\zeta\end{aligned} \tag{3}$$

where, consistent with a thin-shell approximation, terms of order $\zeta/R_{(\ )}$, relative to unity, have been neglected. In Eq. (3), $\{\sigma_\xi,\sigma_\theta\}$, $\{\tau_{\xi\theta}\}$, and $\{\tau_{\xi\zeta},\tau_{\theta\zeta}\}$ represent normal, in-plane shear, and transverse shear stresses, respectively.

By the use of Hooke's law, the stresses can be related to the deformed strain state. If plane sections remain plane, a linear variation of strain across the shell thickness results.

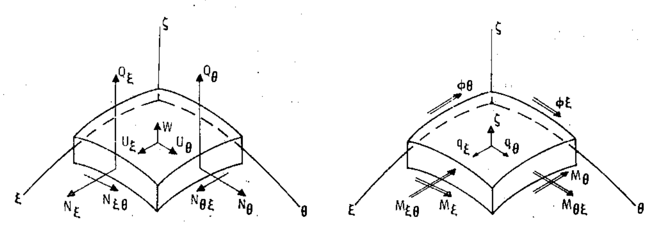

Fig. 2 Sign convention and coordinates.

If the effects of strains and stresses normal to the reference surface are neglected, the elasticity relations at any point in the shell are given for an orthotropic material by the following expressions:

$$\left.\begin{aligned}
\sigma_\zeta &= \frac{E_\xi}{(1-\nu_\xi\nu_\theta)}\{\epsilon_\xi + \nu_\theta\epsilon_\theta + \zeta(\kappa_\xi + \nu_\theta\kappa_\theta) - (\alpha_\xi + \nu_\theta\alpha_\theta)T\} \\
\sigma_\theta &= \frac{E}{(1-\nu_\xi\nu_\theta)}\{\epsilon_\theta + \nu_\xi\epsilon_\xi + \zeta(\kappa_\theta + \nu_\xi\kappa_\xi) - (\alpha_\theta + \nu_\xi\alpha_\xi)T\} \\
\tau_{\xi\zeta} &= G_{\xi\theta}(\epsilon_{\xi\theta} + \zeta\kappa_{\xi\theta}) \\
\tau_{\xi\zeta} &= G_{\xi\zeta}\gamma_{\xi\zeta}, \quad \tau_{\theta\zeta} = G_{\xi\zeta}\gamma_{\theta\zeta}
\end{aligned}\right\} \tag{4}$$

where $\{E_\xi,E_\theta\}$, $\{G_{\xi\theta},G_{\xi\zeta},G_{\theta\zeta}\}$, $\{\nu_\xi,\nu_\theta\}$, and $\{\alpha_\xi,\alpha_\theta\}$ represent Young's moduli, moduli of shear, Poisson's ratios, and coefficients of thermal expansion, respectively. The temperature differential T and all material parameters are permitted to vary in the ξ, θ, and ζ coordinates. The terms $\{\epsilon_\xi,\epsilon_\theta,\epsilon_{\xi\theta}\}$ and $\{\gamma_{\xi\zeta},\gamma_{\theta\zeta}\}$ define membrane and transverse strains of the reference surface, and $\{\kappa_\xi,\kappa_\theta,\kappa_{\xi\theta}\}$ are measures of bending distortion.

Substitution of the stress-strain equations into Eqs. (2) and performance of the appropriate integrations through the thickness yield the following stress-resultant/strain relationships:

$$\left.\begin{aligned}
N_\xi &= B_{10}\epsilon_\xi + B_{30}\epsilon_\theta + B_{11}\kappa_\xi + B_{31}\kappa_\theta - \Theta_{10} \\
N_\theta &= B_{30}\epsilon_\xi + B_{20}\epsilon_\theta + B_{31}\kappa_\xi + B_{21}\kappa_\theta - \Theta_{20} \\
\bar{N}_{\xi\theta} &= B_{40}\epsilon_{\xi\theta} + B_{41}\kappa_{\xi\theta} \\
Q_\xi &= B_{50}\gamma_{\xi\zeta} \\
M_\xi &= B_{11}\epsilon_\xi + B_{31}\epsilon_\theta + B_{12}\kappa_\xi + B_{32}\kappa_\theta - \Theta_{11} \\
M_\theta &= B_{31}\epsilon_\xi + B_{21}\epsilon_\theta + B_{32}\kappa_\xi + B_{22}\kappa_\theta - \Theta_{21} \\
\bar{M}_{\xi\theta} &= B_{41}\epsilon_{\xi\theta} + B_{42}\kappa_{\xi\theta} \\
Q_\theta &= B_{60}\gamma_{\theta\zeta}
\end{aligned}\right\} \tag{5}$$

In the preceding equations, the shell stiffnesses are given in the form

$$B_{mr} = \int \bar{B}_m\zeta^r d\zeta \quad (m = 1,2,3,4,5,6;\ r = 0,1,2) \tag{6}$$

where

$$\bar{B}_1 = \frac{E_\xi}{1-\nu_\xi\nu_\theta}, \quad \bar{B}_2 = \frac{E_\theta}{1-\nu_\xi\nu_\theta}$$

$$\bar{B}_3 = \nu_\xi\bar{B}_2 = \nu_\theta\bar{B}_1, \quad \bar{B}_4 = G_{\xi\theta}$$

$$\bar{B}_5 = G_{\xi\zeta}, \quad \bar{B}_6 = G_{\theta\zeta}$$

but

$$B_{51} = B_{52} = B_{61} = B_{62} = 0$$

The thermal loads are described by

$$\Theta_{mr} = \int \Theta_m T\zeta^r d\zeta \quad (m = 1,2;\ r = 0,1) \tag{7}$$

where

$$\Theta_1 = (\alpha_\xi + \nu_\theta\alpha_\theta)\bar{B}_1, \quad \Theta_2 = (\alpha_\theta + \nu_\xi\alpha_\xi)\bar{B}_2$$

The evaluation of the stiffness quantities is based on the selected reference surface ($\zeta = 0$). In general, it is not possible to simplify these expressions for orthotropic shells having both varying meridional and circumferential stiffness properties. However, for the case where the stiffness properties do not vary in the circumferential direction, it is possible to select the reference surface ($\zeta = 0$) such that stiffness quantities B_{m1} described previously would vanish. For multilayer shells, the integration of Eqs. (6) and (7) must be taken layer by layer through the thickness because of the discontinuities caused by different properties of each layer.

Strain-Displacement Relations

The strains in Eq. (5) may be defined in terms of displacements and rotations by the following expressions. The membrane strains of the reference surface are given by

$$\left.\begin{aligned}
\epsilon_\xi &= \frac{1}{a}\left[\frac{\partial U_\xi}{\partial\xi} + \omega_\xi W\right] \\
\epsilon_\theta &= \frac{1}{a}\left[\frac{1}{\rho}\frac{\partial U_\theta}{\partial\theta} + \gamma U_\xi + \omega_\theta W\right] \\
\epsilon_{\xi\theta} &= \frac{1}{2a}\left[\frac{1}{\rho}\frac{\partial U_\xi}{\partial\theta} + \frac{\partial U_\theta}{\partial\xi} - \gamma U_\theta\right]
\end{aligned}\right\} \tag{8}$$

where U_ξ, U_θ, and W are displacements in the ξ, θ, and ζ directions, respectively.

The change of angle between the reference surface and its normal is measured by transverse strains given by

$$\left.\begin{aligned}
\gamma_{\xi\zeta} &= \Phi_\xi - \frac{1}{a}\left[-\frac{\partial W}{\partial\xi} + \omega_\xi U_\xi\right] \\
\gamma_{\theta\zeta} &= \Phi_\theta - \frac{1}{a}\left[-\frac{1}{\rho}\frac{\partial W}{\partial\theta} + \omega_\theta U_\theta\right]
\end{aligned}\right\} \tag{9}$$

where Φ_ξ, Φ_θ are the rotations of the normal to the reference surface. Since the Kirchhoff hypothesis that normals remain normal will not be assumed here, the transverse shear strains do not, in general, vanish, and the rotations Φ_ξ, Φ_θ become unknown problem variables.

The surface bending distortion terms from Sanders' theory are given by

$$\left.\begin{aligned}
\kappa_\xi &= (1/a)\,(\partial\Phi_\xi)/\partial\xi \\
\kappa_\theta &= \frac{1}{a}\left[\frac{1}{\rho}\frac{\partial\Phi_\xi}{\partial\theta} + \gamma\Phi_\xi\right] \\
\kappa_{\xi\theta} &= \frac{1}{2a}\left[\frac{1}{\rho}\frac{\partial\Phi_\xi}{\partial\theta} + \frac{\partial\Phi_\theta}{\partial\xi} - \gamma\Phi_\theta + \frac{1}{2a}(\omega_\xi - \omega_\theta)\times\left(\frac{1}{\rho}\frac{\partial U_\xi}{\partial\theta} - \frac{\partial U_\theta}{\partial\xi} - \gamma U_\theta\right)\right]
\end{aligned}\right\} \tag{10}$$

The substitution of Eqs. (8–10) into constitutive equations, Eq. (5), and the substitution of the expressions that result into the shell equilibrium equations yield a tenth-order system of partial differential equations (five equations of second order) in terms of the variables U_ξ, $U_\theta W$, Φ_ξ, and Φ_θ.

Fourier Analysis

A Fourier analysis technique will be used in order to permit the separation of variables and to yield a more tractable set of shell equations. The procedure involves expansion of the loads and solution variables in Fourier series. Since the shell stiffness parameters are permitted to vary in the circumferential direction, the stiffness quantities will also be represented by Fourier series. In order to simplify the presentation, the shells considered will be assumed to have circumferential symmetry of shell stiffness about $\theta = 0$. The expansions consistent with this assumption are given by

$$B_{mr} = E_0 h_0 \sum_{j=0}^{\infty} \alpha_r b_j^{(mr)}(\xi)\cos j\theta \tag{11}$$

$$(m = 1,2,3,4,5,6;\ r = 0,1,2)$$

where

$$\alpha_0 = 1,\ \alpha_1 = h_0,\ \alpha_2 = h_0^2$$

The quantities E_0, h_0 represent reference Young's modulus and thickness levels, respectively, which are introduced to provide nondimensional Fourier coefficients. In Eq. (11), the notation (mr) represents a superscript and not an exponent.

Solutions for the shell-field equations are sought in the form

$$\left.\begin{aligned} U_\xi &= \frac{a\sigma_0}{E_0}\sum_{n=0}^{\infty} u_n(\xi)\cos n\theta \\ U_\theta &= \frac{a\sigma_0}{E_0}\sum_{n=1}^{\infty} v_n(\xi)\sin n\theta \\ W &= \frac{a\sigma_0}{E_0}\sum_{n=0}^{\infty} w_n(\xi)\cos n\theta \\ \Phi_\xi &= \frac{\sigma_0}{E_0}\sum_{n=0}^{\infty} \phi_n(\xi)\cos n\theta \\ \Phi_\theta &= \frac{\sigma_0}{E_0}\sum_{n=1}^{\infty} \bar{\phi}_n(\xi)\sin n\theta \end{aligned}\right\} \quad (12)$$

where, to be consistent with Eq. (11) and the assumed stiffness symmetry conditions, the external loads are expanded in the form

$$\left.\begin{aligned} q_\xi &= \frac{\sigma_0 h_0}{a}\sum_{n=0}^{\infty} p_{\xi n}(\xi)\cos n\theta \\ q_\theta &= \frac{\sigma_0 h_0}{a}\sum_{n=1}^{\infty} p_{\theta n}(\xi)\sin n\theta \\ q_\zeta &= \frac{\sigma_0 h_0}{a}\sum_{n=0}^{\infty} p_{\zeta n}(\xi)\cos n\theta \end{aligned}\right\} \quad (13)$$

where σ_0 is a reference stress level. Expansions for the temperature distributions may be described in a similar manner; however, since the thermal coefficient and Young's modulus can vary in the circumferential direction, it will be more convenient to expand the total thermal load in Fourier series as follows:

$$\Theta_{m0} = \sigma_0 h_0 \sum_{n=0}^{\infty} t_{mn}{}^T \cos n\theta,\ \Theta_{m1} = \frac{\sigma_0 h_0^3}{a}\sum_{n=0}^{\infty} \bar{t}_{mn}{}^T \cos n\theta \quad (14)$$

where the Fourier coefficients $t_{mn}{}^T$, $\bar{t}_{mn}{}^T$ are evaluated by the Fourier-Euler inversion formula.

The Fourier series expressions for forces and moments in Eq. (5) are obtained by using Eqs. (8–12) with the proper application of appropriate trigonometric identities.[6] The stress-resultant series in terms of the Fourier coefficients of the displacement, rotation, and stiffness parameters is given by

$$\left.\begin{aligned} N_\xi &= \sigma_0 h_0 \sum_{n=0}^{\infty}\left[\sum_{j=0}^{\infty}\{A_{10}e_{\xi j} + A_{30}e_{\theta j} + \lambda A_{11}\kappa_{\xi j} + \lambda A_{31}\kappa_{\theta j}\} - t_{1n}{}^T\right]\cos n\theta \\ N_\theta &= \sigma_0 h_0 \sum_{n=0}^{\infty}\left[\sum_{j=0}^{\infty}\{A_{30}e_{\xi j} + A_{20}e_{\theta j} + \lambda A_{31}\kappa_{\xi j} + \lambda A_{21}\kappa_{\theta j}\} - t_{2n}{}^T\right]\cos n\theta \\ \bar{N}_{\xi\theta} &= \sigma_0 h_0 \sum_{n=1}^{\infty}\left[\sum_{j=0}^{\infty}\{A_{40}e_{\xi\theta j} + \lambda A_{41}\kappa_{\theta\xi j}\}\right]\sin n\theta \\ Q_\xi &= \sigma_0 h_0 \sum_{n=0}^{\infty}\left[\sum_{j=0}^{\infty} A_{50}\gamma_{\xi j}\right]\cos n\theta \\ M_\xi &= \frac{\sigma_0 h_0^3}{a}\sum_{n=0}^{\infty}\left[\sum_{j=0}^{\infty}\left\{\frac{A_{11}}{\lambda}e_{\xi j} + \frac{A_{31}}{\lambda}e_{\theta j} + A_{12}\kappa_{\xi j} + A_{32}\kappa_{\theta j}\right\} - \bar{t}_{1n}{}^T\right]\cos n\theta \\ M_\theta &= \frac{\sigma_0 h_0^3}{a}\sum_{n=0}^{\infty}\left[\sum_{j=0}^{\infty}\left\{\frac{A_{31}}{\lambda}e_{\xi j} + \frac{A_{21}}{\lambda}e_{\theta j} + A_{32}\kappa_{\xi j} + A_{22}\kappa_{\theta j}\right\} - \bar{t}_{2n}{}^T\right]\cos n\theta \\ M_{\xi\theta} &= \frac{\sigma_0 h_0^3}{a}\sum_{n=1}^{\infty}\left[\sum_{j=0}^{\infty}\left\{\frac{A_{41}}{\lambda}e_{\xi\theta j} + A_{42}\kappa_{\xi\theta j}\right\}\right]\sin n\theta \\ Q_\theta &= \sigma_0 h_0 \sum_{n-0}^{\infty}\left[\sum_{j=0}^{\infty} A_{50}\gamma_{\theta j}\right]\cos n\theta \end{aligned}\right\} \quad (15)$$

where Fourier coefficients of strain $e_{(1)}$ and bending distortion $\kappa_{(1)}$ are given by

$$\left.\begin{aligned} e_{\xi j} &= [u'_j + \omega_\xi w_j],\ e_{\theta j} = [(j/\rho)v_j + \gamma u_j + \omega_\theta w_j] \\ e_{\xi\theta j} &= \tfrac{1}{2}\,[-(j/\rho)u_j + v'_j - \gamma v_j] \\ \kappa_{\xi j} &= \phi'_j,\ \kappa_{\theta j} = [(j/\rho)\phi_j + \gamma\phi_j] \\ \kappa_{\xi\theta j} &= \tfrac{1}{2}[-(j/\rho)\phi_j + \phi'_j - \gamma\phi_j + \tfrac{1}{2}(\omega_\theta - \omega_\xi)\{(j/\rho)u_j + v'_j + \gamma v_j\}] \\ \gamma_{\xi j} &= \phi_j - [-w'_j + \omega_\xi u_j],\ \gamma_{\theta j} = \phi_j - [(j/\rho)w_j + \omega_\theta v_j] \end{aligned}\right\} \quad (16)$$

and

$$\lambda = h_0/a$$

Derivatives with respect to the meridional coordinate are denoted by primes [i.e., $\partial/\partial\xi(\ ) = (\ )'$]. The stiffness recursion relationships A_{mr} in Eq. (15) are functions of the Fourier coefficients of stiffness $b_j{}^{(mr)}$ and the Fourier indices n, j. (The indices have been dropped for ease of presentation.) The stiffness recursion expressions A_{mr} are given in the form

$$A_{mr} = \tfrac{1}{2}\{\bar{\delta}b_{(n+j)}{}^{(mr)} + [1 - \eta^2(j-r) + \eta(n)]b_{|n-j|}{}^{(mr)}\} \quad (m = 1,2,3,4,5,6;\ r = 0,1,2) \quad (17)$$

where the symbolic function $\eta(l)$ is denfied as

$$\eta(l) = \begin{cases} -1 & (l < 0) \\ 0 & (l = 0) \\ 1 & (l > 0) \end{cases} \tag{18}$$

and

$$\bar{\delta} = \begin{cases} 1 & (m = 1,2,3,5) \\ -1 & (m = 4,6) \end{cases} \tag{19}$$

As implied earlier, the Fourier expansions in the previous development are not the most general form that can exist. If the assumption of circumferential symmetry with respect to loads (q_ξ, q_ζ) and stiffness properties were relaxed, it would then be necessary to augment all the cosine series expansions in Eqs. (11–14) by appropriate sine series expansions. Similarly, the antisymmetric expansions about $\theta = 0$ (i.e., q_θ, U_θ) would be augmented by corresponding cosine series.

For the general case, the formulation would proceed in a manner similar to the development previously presented. In the next section, the stress-resultant series expressions will be used to reduce the shell equations to a form that will permit efficient numerical solution.

Reduced Shell Differential Equations

Substitution of the series expressions, Eq. (15), into the shell equilibrium equations, Eq. (2), yields five infinite series expressions in the circumferential coordinate relating the Fourier coefficients of the displacement and rotation variables: u_j, v_j, w_j, ϕ_j, and $\bar{\phi}_j$. Application of the appropriate orthogonality conditions to these equilibrium series expressions permits the elimination of the circumferential coordinate θ. For computational purposes, truncation of the Fourier series after K terms results in a system of $5K$ ordinary differential equations relating the $5K$ unknown Fourier coefficients of displacement and rotation. The resulting shell equations are described as follows:

$$\left.\begin{aligned}
&\sum_{j=0}^{K-1}\sum_{l=1}^{5}[f_{5k+1,5j+l}\, z''_{5j+l} + g_{5k+1,5j+l}\, z'_{5j+l} + h_{5k+1,5j+l}\, z_{5j+l}] = p_{1k}\\
&\sum_{j=0}^{K-1}\sum_{l=1}^{5}[f_{5k+2,5j+l}\, z''_{5j+l} + g_{5k+2,5j+l}\, z'_{5j+l} + h_{5k+2,5j+l}\, z_{5j+l}] = p_{2k}\\
&\sum_{j=0}^{K-1}\sum_{l=1}^{5}[f_{5k+3,5j+l}\, z''_{5j+l} + g_{5k+3,5j+l}\, z'_{5j+l} + h_{5k+3,5j+l}\, z_{5j+l}] = p_{3k}\\
&\sum_{j=0}^{K-1}\sum_{=1}^{5}[f_{5k+4,5j+l}\, z''_{5j+l} + g_{5k+4,5j+l}\, z'_{5j+l} + h_{5k+4,5j+l}\, z_{5j+l}] = p_{4k}\\
&\sum_{j=0}^{K-1}\sum_{=1}^{5}[f_{5k+5,5j+l}\, z''_{5j+l} + g_{5k+5,5j+l}\, z'_{5j+l} + h_{5k+5,5j+l}\, z_{5j+l}] = p_{5k}
\end{aligned}\right\} \tag{20}$$

where $k = 0,1,2,3,\ldots,K-1$ and

$$z_{5j+1} = u_j,\ z_{5j+2} = v_j,\ z_{5j+3} = w_j,\ z_{5j+4} = \phi_j,\ z_{5j+5} = \bar{\phi}_j \tag{21}$$

The $f_{(\ )}$, $g_{(\ )}$, and $h_{(\ )}$ coefficients are functions relating the shell geometrical parameters, stiffness quantities, and Fourier indices. The quantities $p_{(\ )}k$ are functions of the known Fourier coefficients of the applied external and thermal loads. The coefficients f, g, h, and p are defined in detail in Ref. 6.

It should be noted that the form of the preceding equations is more complicated than that obtained in Ref. 1 for the analysis of shells of revolution. This complexity arises from the fact that circumferential variations in stiffness prevents the equilibrium equations from being uncoupled into separate sets for each Fourier index n.

The preceding $5k$ equations can be conveniently written in the following matrix form:

$$Fz'' + Gz' + Hz = p \tag{22}$$

where F, G, H, and z,p are $[5K \times 5K]$ and $[1 \times 5K]$ matrices, respectively.

Boundary Conditions

Consistent with Sanders' theory, the proper form of the boundary conditions to be prescribed at an edge where ξ = constant (i.e., boundaries $s = 0$, $s = s^*$) is given by specifying the following:

$$\begin{aligned} &N_\xi \text{ or } U_\xi\\ &\hat{N}_{\xi\theta} \text{ or } U_\theta\\ &Q_\xi \text{ or } W\\ &M_\xi \text{ or } \Phi_\xi\\ &\bar{M}_{\xi\theta} \text{ or } \Phi_\theta \end{aligned}$$

where

$$\hat{N}_{\xi\theta} = \bar{N}_{\xi\theta} + (1/2a)(\omega_\theta - \omega_\xi)\bar{M}_{\xi\theta}$$

In a format similar to that used in Ref. 1, the combinations of boundary conditions that can be prescribed are conveniently expressed by use of the matrix equation

$$\bar{\Omega}y + \bar{\Lambda}\bar{z} = \bar{l} \tag{23}$$

where

$$\bar{y} = \begin{Bmatrix} N_\xi \\ N_{\xi\theta} \\ Q_\xi \\ M_\xi \\ \bar{M}_{\xi\theta} \end{Bmatrix},\quad \bar{z} = \begin{Bmatrix} U_\xi \\ U_\theta \\ W \\ \Phi_\xi \\ \Phi_\theta \end{Bmatrix},\quad \bar{l} = \begin{Bmatrix} l_1 \\ l_2 \\ l_3 \\ l_4 \\ l_5 \end{Bmatrix}$$

$\bar{\Omega}$, $\bar{\Delta}$ are the appropriate diagonal matrices used for specification of the kind of boundary, and l is a given column matrix.

Substitution of series expansions, Eqs. (12) and (15), into Eq. (23) and application of appropriate orthogonality relationships allow the boundary conditions to be described in terms of Fourier coefficients of displacements and rotation variables using the following matrix form:

$$\Omega Rz' + (\Lambda + \Omega S)z = l - \Omega t^* \tag{24}$$

where elements of R, S, and t^* matrices are described in Ref. 6.

The form of Eq. (24) must be modified if the shell has a pole (i.e., $r = 0$), because the coefficients of the differential equations become singular for this case. The conditions can be obtained using a limiting process similar to that described by Greenbaum.[7]

Discontinuity and branching shell problems will not be discussed here. However, these situations can be handled with a similar procedure, illustrated in Ref. 1, which involves the introduction of appropriate compatibility expressions relating displacements and forces across a discontinuity.

The reduced equilibrium equations and boundary conditions are now in a convenient form for numerical solution.

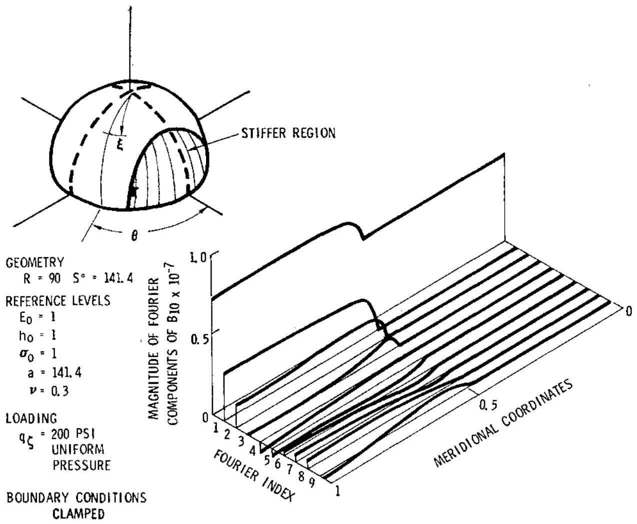

Fig. 3 Fourier components of the membrane stiffness distribution for the hemispherical shell problem.

Numerical Solution

The matrix form of the shell-field equations, Eq. (22), and the boundary conditions, Eq. (24), is similar to the equations developed in Ref. 1. This similarity will permit utilization of the same numerical procedure for solution of the shell equations.

When use is made of the finite-difference forms presented in Ref. 1, the shell equations can be expressed by the following sets of algebraic equations relating the solution matrix z_i ($i = 0,1,2,\ldots N$):

$$\begin{aligned} A_0z_1 + B_0z_0 &= g_0 \\ A_iz_{i+1} + B_iz_i + C_iz_{i+1} &= g_i \\ B_Nz_N + C_Nz_{N+1} &= g_N \end{aligned} \qquad (25)$$

where the matrices A, B, C, and g are similar in form to Eqs. (67–71) of Ref. 1.

Equation (25) will be solved with a direct matrix elimination technique (Potters' method) which is described in detail in Ref. 1. The basic procedure involves the relating of z_i to z_{i+1} by expressions of the form

$$z_i = -P_iz_{i+1} + x_i(i = 1,2,\ldots N - 1) \qquad (26)$$

where the form of P_i is similar to Ref. 1.

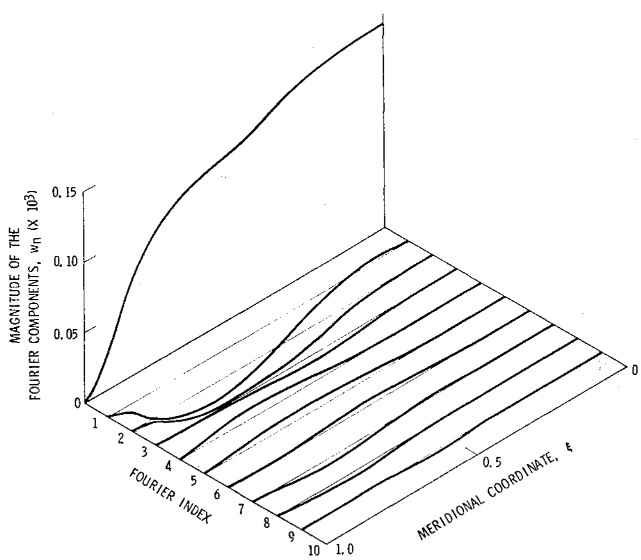

Fig. 4 Fourier component distribution of normal displacements.

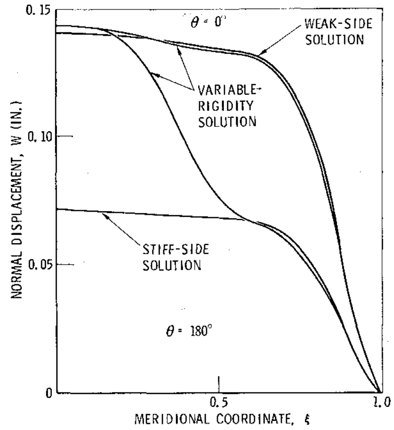

Fig. 5 Meridional distribution of total normal displacements (W) along the plane of symmetry.

From Eq. (26) and the last expression of Eq. (25), the correct value of z_N can be calculated. All the remaining Z_i are calculated in reverse order using Eq. (26) and starting from the calculated value z_N.

It should be noted that for this problem the solution technique described previously involves the matrix inversion of $5K \times 5K$ matrices, with all the Fourier coefficients contributing simultaneously. With the solution matrices z_i known, the forces, moments, stresses, etc. can be evaluated using the expressions developed earlier.

Summary and Discussion of Results

A general analytical and numerical procedure has been presented for the analysis of a class of shells of revolution having arbitrary stiffness distributions. In order to illustrate the proposed approach, the procedure was programed for solution on a digital computer (IBM 7094). In this section, preliminary results obtained from the programed solution for a sample shell problem will be presented. Although limited, these results indicate some interesting phenomena concerning the basic character of the solution procedure.

The problem of a hemispherical shell having a discontinuity in the stiffness distribution was treated. For convenience, the shells were assumed to be isotropie, and the loading was assumed to be uniform internal pressure. The detailed description of this shell configuration, which has a local region of increased stiffness, is illustrated in Fig. 3. Also shown in the figure is a plot of the Fourier coefficients used to describe the extensional stiffness characteristics at meridional locations. For this problem, the Fourier series approximating the discontinuous stiffness distribution was truncated after ten terms. For a uniform pressure loading, individual Fourier coefficient solutions (w_n) for normal deflection are given in Fig. 4 as a function of the meridional coordinate. It can be seen that the general character of the displacement pattern

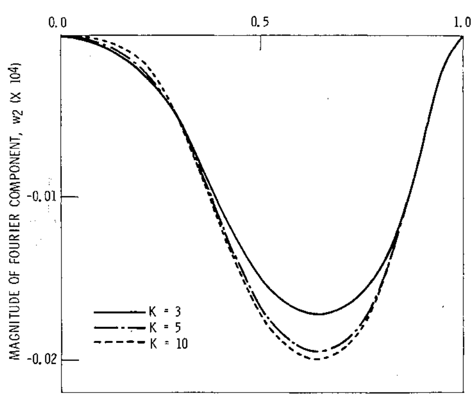

Fig. 6 Meridional distribution of the third Fourier components of the normal displacement (w_2) of solutions for K = 3, 5, and 10.

reflects the Fourier stiffness distribution profile. (A convergence trend is exhibited because the higher-order harmonics of the solution diminish in magnitude.) The total deflection obtained by summing the Fourier contributions is plotted along the plane of symmetry ($\theta = 0\text{–}180°$) in Fig. 5. Also shown are the deflections obtained for axisymmetric shells with uniform stiffness distributions corresponding to the magnitude of the stiff and weak sides. As expected, these results bracket the solution for the variable-rigidity shell.

The same problem was a run with only three and five terms retained in the stiffness series expansions. Variation in the third harmonic of deflection (w_2) along the meridian is plotted in Fig. 6 for the cases where K is set equal to 3, 5, and 10. The higher harmonic coupling effects appear to diminish as more terms are retained in the solution (i.e., as K increases). For economic purposes, the number of integration points used in obtaining the preceding numerical results was $N = 26$.

The basic character of the solutions is illustrated for the relatively coarse grid. Additional studies have shown increased accuracy when a finer finite-difference mesh is taken; this should be considered when the results presented in this paper are interpreted.

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Finite-Element Postbuckling Analysis of Thin Elastic Plates

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A finite-element formulation for determining postbuckling response of thin elastic plates is presented. The method employs an iterative approach to arrive at equilibrium configurations. For each iterate, increments in displacements are estimated by formulating an approximate incremental stiffness matrix and solving the linearized incremental equilibrium equations. The approximate incremental stiffness is determined by combining a geometric stiffness with the stiffness of the unstressed, but displaced, structure. A method of deriving consistent geometric stiffness matrices for two-dimensional plate elements is presented and the principal terms in this matrix are evaluated for a triangular element. Results of the determination of postbuckling response for some typical plates are given.

Nomenclature

$[B]$	= matrix of influence functions specifying increments in element strains for increments in element nodal displacements
$C_{ijkl};[\bar{C}]$	= linear elastic moduli; plane stress elastic constitutive matrix
$[\tilde{D}]$	= matrix of influence functions specifying increments in element middle surface displacements for increments in element nodal displacements
$E_{ij};\Delta E_{ij}$	= Green's strain tensor and its increment
h	= plate thickness
i,j,k,l	= indices with range of three
K_I,K_G,K_E	= incremental, geometric, and element stiffnesses, respectively
$M_{\alpha\beta},N_{\alpha\beta}$	= in-plane stress couples and stress resultants
o	= subscript indicating original configuration
Q_α	= shear stress resultants
$\{r_E\}$	= vector of element nodal displacements, Eq. (13)
$\{\Delta r\};\{\Delta R\}$	= structure nodal displacement increments and nodal force increments
$S_{ij};\Delta S_{ij}$	= Kirchhoff's stress tensor and its increment
$S_0;dS_0$	= surface area and differential surface area
$T_i;\Delta T_i$	= Kirchhoff stress vector and its increment
$[T]$	= corner transformation matrix
T	= superscript indicating transpose
$\bar{u}_i;u_i$	= initial displacements and displacement increments referred to local coordinate system, Fig. 1
$\bar{U}_i;U_i$	= initial displacements and displacement increments referred to global coordinate system, Fig. 1
$\{u\},\{v\},\{w\}$	= vectors of element displacement increments, Fig. 2, in local coordinate system
$\{U\},\{V\},\{W\}$	= vectors of element displacement increments in global coordinate system
x,y,z	= local coordinate system, Fig. 1
X,Y,Z	= global coordinate system, Fig. 1
α,β	= indices with range of two
δ	= indicates virtual variation

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